# ON THE CONVECTIVE INSTABILITY OF FLUIDS IN INTERCONNECTED VERTICAL CHANNELS

## (O KONVEKTIVNOI NEUSTOICHIVOSTI ZHIDKOSTI V SVIAZÁNNYKH VERTIKAL'NYKH KANAIAKH)

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As is known, the convective motion of a fluid heated at the bottom is the result of an equilibrium breakdown caused by the temperature gradient reaching a definite critical value. The equilibrium stability was most thoroughly investigated in relation to a horizontal layer of fluid (the Rayleigh problem, see review in [1]). Ostroumov [2 and 3] had investigated the conditions of convection generation in a vertical circular cylinder. Later investigations dealt with cylinders of other cross sections [4 to 7] and cavities of different forms [8 to 10]. These investigations related to single vertical cylinders and cavities enclosed by heat conducting solids, or with specified boundary conditions.

Of great interest is the problem of convective instability of a fluid contained in a system of cavities subject to thermal interaction via a heat conducting solid, and in particular, in a system of vertical channels. Such problems have, apparently, not been analyzed so far. This paper gives an exact solution of the equilibrium stability problem for the case of two parallel vertical flat channels, separated by a solid mass. An approximate solution is also presented for the problem of two vertical cylindrical channels of circular cross section in a solid. The critical Rayleigh number, which determines the limit of instability, is derived in terms of thermal conductivity of the fluid and solid, and of the distance between the two channels.

**1. Flat channels.** Two vertical parallel flat fluid layers (each layer is $2h$ thick, the distance between their centers is $2d$, and the $z$-axis points vertically upwards) are provided in a homogeneous heat conducting solid (Fig.1). Under equilibrium conditions the fluid is motionless and the temperature gradient is constant and vertical

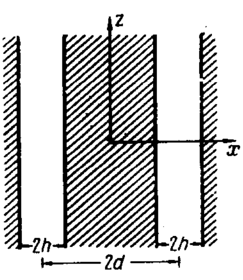


Fig. 1

$$\mathbf{v}_0 = 0, \qquad \frac{dT_0}{dz} = -A \tag{1.1}$$

We shall consider two-dimensional perturbations of equilibrium of the form

$$v_x = v_y = 0, \; v_z = v(x), \; T = T(x), \; \nabla p = 0 \tag{1.2}$$

Here $\mathbf{v}$, $T$ and $\rho$ are respectively the perturbations of velocity, temperature and pressure. The dependence of normal perturbations on time is subject to the $\exp(-\sigma t)$ law, where $\sigma$ is a real decrement (for heating from below) [11]. At the limit of stability $\sigma = 0$, i.e. neutral perturbations are stationary. We write the equations of neutral perturbations in a dimensionless form (with primes denoting differentiation with respect to $x$)

$$v'' + RT = 0, \qquad T'' + v = 0, \qquad T_m'' = 0 \qquad \left(R = \frac{g\beta A h^4}{\nu\chi}\right) \tag{1.3}$$

Here $v$, $T$ are respectively the dimensionless perturbations of velocity and temperature of the fluid, $T_m$ is the solid mass temperature perturbation, $R$ is the Rayleigh number, $g$ the acceleration of gravity, and $\beta$, $\nu$, $\chi$ are respectively the coefficients of the fluid thermal expansion, kinematic viscosity, and thermal diffusivity. As units of length. velocity and temperature we select $h$, $\chi/h$ and $Ah$.

At the liquid – solid interfaces the velocity must become zero, and the continuity conditions of temperature and heat flow must be fulfilled. Inasmuch as the conductivity equation in the solid is expressed by $T_m'' = 0$, the temperature in it must be linearily dependent on the coordinate.

The postulation of boundedness of temperature perturbation $T_m$ with $x \to \pm\infty$ leads to the conclusion that the temperature in the outer regions of the solid is constant, and that there is no horizontal flow of heat in these regions. On the other hand, a horizontal heat flow may exist in the layer between the two channels resulting in a thermal interaction between these. Thus, boundary conditions of Equations (1.3) are

$$\begin{aligned} &v = 0, \quad T = T_m, \quad \lambda T' = T_m' \qquad \text{for } x = x_1 \\ &v = 0, \quad T' = 0 \qquad \text{for } x = x_2 \\ &\left(x_1 = \pm\frac{d-h}{h}, \quad x_2 = \pm\frac{d+h}{h}\right) \end{aligned} \tag{1.4}$$

Here $\lambda = \varkappa/\varkappa_m$, and $\varkappa$ and $\varkappa_m$ are the thermal conductivities of the liquid and solid respectively, $x_1$ and $x_2$ are the inner and outer boundaries of the right-hand (plus sign) and on the left-hand (minus sign) side channels. Further to this, the condition of closed flow stream must be fulfilled in the case of free convection in a two-channel system

$$\int_{x_2}^{x_1} v_- \, dx + \int_{x_1}^{x_2} v_+ \, dx = 0 \tag{1.5}$$

where $v_+$ and $v_-$ are the velocities in the right- and left-hand side channels respectively. It is assumed that the two channels are interconnected at some distance at the top and bottom, and that the fluid can pass from one channel to the other, so that the rate of flow across the section of one channel may be different from zero.

Problem (1.3) to (1.5) has even and odd solutions with respect to $x$.

We shall first consider the odd type solutions. In this case the temperature in the layer of solid between the channels is $T_m = cx$. For the determination of velocity in the fluid it will be convenient to eliminate $T$ from Equations (1.3)

$$v^{\mathrm{IV}} + Rv = 0 \tag{1.6}$$

The general solution of Equation (1.6) has the form

$$v = A \sin rx + B \cos rx + C \sinh rx + D \cosh rx \tag{1.7}$$

where $r = R^{\frac{1}{4}}$ . The temperature in the fluid is then

$$T = r^{-2}(A \sin rx + B \cos rx - C \sinh rx - D \cosh rx) \tag{1.8}$$

Boundary conditions (1.4) lead to a system of homogeneous equations for the determination of constants $A$, $B$, $C$, $D$ and $c$ (condition of closed flow (1.5) is automatically fulfilled in the case of an odd solution). The condition of this system solvability yields an expression from which the critical value of the Rayleigh number is determined

$$\frac{\tan 2r + \tanh 2r}{r(\sec 2r \operatorname{sech} 2r - 1)} = \lambda x_1 \tag{1.9}$$

Having determined the constants of integration, we find the velocity and temperature distribution

$$v = \pm\left[\frac{\cos r(x_2 - x) - \cosh r(x_2 - x)}{\cos r(x_2 - x_1) - \cosh r(x_2 - x_1)} - \frac{\sin r(x_2 - x) + \sinh r(x_2 - x)}{\sin r(x_2 - x_1) + \sinh r(x_2 - x_1)}\right]$$

$$T = \pm\frac{1}{r^2}\left[\frac{\cos r(x_2 - x) + \cosh r(x_2 - x)}{\cos r(x_2 - x_1) - \cosh r(x_2 - x_1)} - \frac{\sin r(x_2 - x) - \sinh r(x_2 - x)}{\sin r(x_2 - x_1) + \sinh r(x_2 - x_1)}\right] \tag{1.10}$$

$$T_m = \frac{2\lambda(1 - \cos 2r \cosh 2r)}{r(\cos 2r - \cosh 2r)(\sin 2r + \sinh 2r)}\, x$$

The plus and minus signs refer to the left- and right-hand side channels respectively. The solution (1.10) amplitude remains arbitrary in view of the problem homogeneity.

In the case of an even solution the temperature of the intermediate solid layer is constant, $T_m =$ const . Constants of integration are derived from the boundary conditions (1.4) and the condition of closed flow (1.5). In view of the velocity being even ($v_+ = v_-$) , the latter must be fulfilled separately in each of the channels. The fluid velocity and temperature are defined by Formulas (1.10) with a plus sign for each of the two channels, but with different values of the critical number $r$ which in this case is determined by the characteristic relationship

$$\tan^2 r - \tanh^2 r = 0 \tag{1.11}$$

The temperature in the intermediate solid layer is

$$T_m = \frac{2(\sin 2r \cosh 2r + \cos 2r \sinh 2r)}{r^2(\cos 2r - \cosh 2r)(\sin 2r + \sinh 2r)} \tag{1.12}$$

Thus, Equations (1.9) and (1.11) determine the spectrum of critical Rayleigh numbers with respect to $x$ for the even and odd type of flow. It will be seen from (1.9) that the critical numbers $r$ which correspond to odd levels depend on one parameter $\lambda x_1$ which defines the thermal relationship of the two channels. For example, large values of this parameter correspond of the two channels. For example, large values of this parameter correspond to a weak interaction between channels (great thickness of the intermediate layer, or its low thermal conductivity).

It will be seen from (1.11) that in the case of an even solution, the critical numbers do not depend on the intermediate layer parameters. This is due to the temperature there being, in this case, constant and to the absence of any horizontal heat flow, i.e. there is no thermal interaction between the channels. In each of the channels an "autonomous" circulation is originated with a zero flow rate across its section. Critical values of the Rayleigh number $r$ coincide, quite naturally, with values which define the equilibrium stability limit in a single flat channel [6]. Under these conditions the even and odd kinds of motion with respect to the middle of the channel correspond to the two groups of solutions of Equation (1.11)

$$\tan r = \pm\tanh r$$

The odd ($r_1$, $r_3$,...) and even ($r_2$, $r_4$,...) lower levels of the spectrum of critical numbers $r$ are shown on Fig.2 as functions of $\lambda x_1$. It will be seen from this graph that for $\lambda x_1 \to \infty$ the "odd" critical values of the Rayleigh number decrease (decreasing stability). Of the greatest interest is the lower level of $r_1$ which actually determines the convection threshold. With $\lambda x_1 \to \infty$, i.e. with the weakening of channel interaction, the lowest critical number $r_1$ tends to zero, and the equilibrium becomes absolutely unstable. We note that with the weakening of interaction ($\lambda x_1 \to \infty$) the odd type motions become practically "autonomous", and that consequently the even and odd levels corresponding to motions with an equal number of nodes ($r_2$ and $r_3$, $r_4$ and $r_5$, etc.) are drawn together.

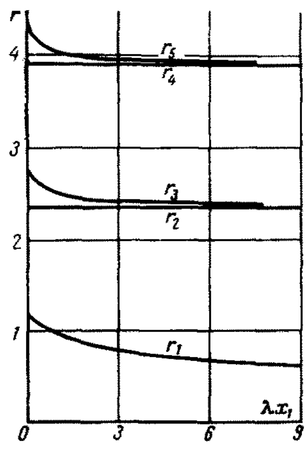


Fig. 2

At the limit of decreasing distance between the two channels ($\lambda x_1 \to 0$) the critical numbers are those of a single channel and correspond to motions with the velocity node at the channel center.

We may note in conclusion that the solutions derived in this Section are exact stationary solutions of the nonlinear convection equations.

**2. Cylindrical channels.** We shall consider now two vertical circular cylindrical channels of the same radius $\rho$ spaced at a distance $2d$ between their axes, and surrounded by a heat conducting solid mass. We consider equilibrium perturbations defined by

$$v_x = v_y = 0, \qquad v_z = v(x, y), \qquad T = T(x, y), \qquad \nabla p = 0 \qquad (2.1)$$

and instead of (1.3) obtain Equations

$$\Delta v + RT = 0,\ \Delta T + v = 0,\ \Delta T_m = 0 \quad \left(\Lambda = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad R = \frac{g\beta A\rho^4}{\nu\chi}\right) \ (2.2)$$

All parameters of Equations (2.2) are dimensionless, with the cylinder radius $\rho$ as unit. The position of axes in the horizontal plane is shown on Fig.3.

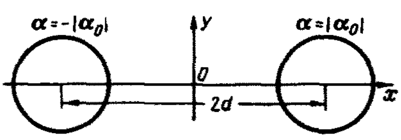


Fig. 3

We introduce bipolar coordinates ($\alpha$,$\beta$) defined by relations

$$x = \frac{a \sinh\alpha}{\cosh\alpha + \cos\beta}, \quad y = \frac{a \sin\beta}{\cosh\alpha + \cos\beta} \quad (2.3)$$

($2\alpha$ is the distance between poles). At the fluid – solid interface the usual conditions for velocity and temperature must be fulfilled

$$v = 0, \qquad T = T_m, \qquad \lambda \frac{\partial T}{\partial \alpha} = \frac{\partial T_m}{\partial \alpha} \qquad \text{for } \alpha = \pm |\alpha_0| \tag{2.4}$$

(the plus and minus signs correspond to the right- and left-hand side channels).

We shall limit ourselves to finding an approximate solution which would determine the emergence of convection (the lower odd level). With this in view we approximate the velocity by a polynomial which satisfies boundary condition

$$v^* = v_0 \left[1 - (x - d)^2 - y^2\right] \tag{2.5}$$

Here $v_0$ denotes an arbitrary (because of the problem homogeneity) amplitude of motion, with different signs for the right- and left-hand side channels.

The temperature $T_m$ in a solid mass is a harmonic function odd with respect to $x$ (i.e. also with respect to $\alpha$ ) which vanishes at infinity (for $\alpha \to 0$) and is periodic with respect to $\beta$

$$T_m = c_0\alpha + c_1 \sinh \alpha \cos \beta + c_2 \sinh 2\alpha \cos 2\beta + \ldots \tag{2.6}$$

The fluid temperature is also approximated by a polynomial of the form

$$T^* = A + B\,(x - d) + C\,(x - d)^2 + Cy^2 \tag{2.7}$$

The constants of this expression of $T^*$ will be determined from boundary conditions and the requirement that $T^*$ (in accordance with Galerkin's method) must approximately satisfy the termal conductivity equation

$$\int (\Delta T^* + v^*)\, T^*\, dS = 0 \tag{2.8}$$

(integration is carried out over the channel cross sections). Expanding $T^*$ into a Fourier series with respect to $\beta$ , and limiting this expansion, as well as that of (2.6) to the first two harmonics, we obtain from the temperature boundary conditions and the integral condition (2.8) five relationships

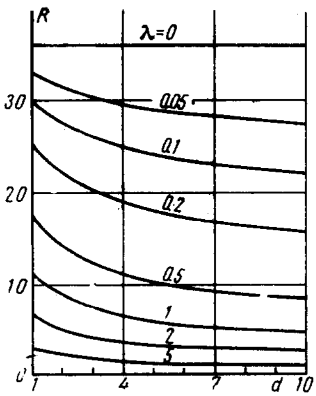


Fig. 4

$$A = -(1 + f)\,C, \qquad B = (2\lambda\alpha_0 - f)\, e^{\alpha_0} C$$

$$c_0 = -2\lambda C$$

$$c_1 = 2\,(f - 2\lambda\alpha_0)\,C, \qquad C = -v_0 \frac{2 + 3f}{12\,(1 + 2f)}$$

which determine the five constants $A$, $B$, $C$, $c_0$ and $c_1$. We find, as a result

$$f(\alpha_0, \lambda) = 2\lambda \left(\alpha_0 + \frac{e^{-\alpha_0}}{\cosh \alpha_0 + \lambda \sinh \alpha_0}\right) \tag{2.9}$$

In this manner we have derived approximate expressions of temperatures

$T^*$ and $T^*_s$ , corresponding to the velocity approximation (2.5).

Substituting into the first of Equations (2.2) the approximations of $v^*$ and $T^*$, multiplying it by $v^*$, and integrating over the cross sections of channels, we derive the condition of solvability of system (2.2), from which the Rayleigh number critical value can be determined

$$R = -\int \Delta v^* v^* \, dS \Big/ \int T^* v^* \, dS \qquad (2.10)$$

After computation, we find

$$R = \frac{144\,(1 + 2f)}{(2 + 3f)^2} \qquad (2.11)$$

where $f$ is a known function of $\lambda$ and $\alpha_0$ (see (2.9)). Formula (2.11) makes it possible to express the critical Rayleigh number $R$ as a function of the ratio the fluid and solid mass thermal conductivities $\lambda = \varkappa/\varkappa_s$ and of the distance of the channel axes (in units of their radius) $2d = 2\cosh \alpha_0$.

Fig.4 shows curves depicting the dependence of $R$ on the dimensionless distance $d$ for several values of $\lambda$ . It will be seen that the most stable equilibrium obtains for $\lambda = 0$ (infinite thermal conductivity of the solid). In this case the critical Rayleigh number is at its maximum and independent of the distance between channels. With increasing $d$ and $\lambda$ (i.e. with a weakening thermal interaction between channels) the critical number decreases.

Reverting to Formula (2.11), we note that the critical Rayleigh number with the approximation considered here, is, as a matter of fact, determined by one parameter, namely $f$ . This parameter can be given a physical meaning by relating it to the effective value of the dimensionless Biot number $b$ which we shall define by the heat flow and temperature averaged over the channel boundary

$$b = -\left\langle \left(\frac{\partial T}{\partial n}\right)_{\alpha_0} \right\rangle \Big/ \langle (T)_{\alpha_0} \rangle \qquad (2.12)$$

The sign $\langle\rangle$ denotes here the averaging over the boundary. Substituting $T^*$ we obtain $b = 2/f$ . Thus parameter $f$ decreases with the increase of heat transfer between fluid and solid, i.e. with the increase of interaction between channels. This value is the analog of the interaction parameter $\lambda x_1$ in the case of flat channels (see Section 1). For large distances $(d \gg 1$, $\alpha_0 \gg 1)$

$$f = 2\lambda\alpha_0 = 2\lambda \ln 2d \qquad (2.13)$$

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